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A FEASIBILITY CONJECTURE RELATED TO THE HIRSCH CONJECTURE

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I. Introduction

A very important unsolved problem in linear programming today is the validity or nonvalidity of the conjecture which follows:

Hirsch Conjecture: Let X and Y be feasible bases to the linear programming problem

$$(1) \qquad \begin{gathered} Y \geq 0 , \; X \geq 0 \\ IY + BX = b \end{gathered}$$

and let m be the rank of this system. Then the conjecture is that there exists a sequence of m feasible pivot operations that transforms system (1) into

$$(2) \qquad \begin{gathered} Y \geq 0 , \; X \geq 0 \\ B^{-1}Y + IX = B^{-1}b \end{gathered}$$

That is to say a sequence such that each of the $m - 1$ intermediate bases is feasible. In order to attack this problem, consider the following possibility.

Suppose there exists a pair of variables x_j , $y_{k(j)}$ such that if x_j is introduced into the basis in (1), $y_{k(j)}$ drops out and, in addition, every feasible solution must contain at least one of these variables at a non-zero value. It follows that if x_j is introduced into the basis, dropping $y_{k(j)}$, then introducing any further sequence of variables not involving $y_{k(j)}$ into the basis cannot drop x_j . This essentially reduces the problem of finding m feasible pivots

to one of finding $m - 1$ feasible pivots. The same result follows if there is a pair $x_{\ell(j)}$, y_j in (2) with these properties, since any sequence of feasible pivots going from (2) to (1) is the reverse of a sequence of feasible pivots going from (1) to (2).

A conjecture proposed by G. B. Dantzig, which (as we have just noted) would, if true, imply the Hirsch conjecture states that there always exists such a pair, provided that the basis for X is non-degenerate. This latter conjecture is actually false in general as will be shown by a counter example. Except for this example, all others (and there were many) selected at random satisfied the conjecture. It is felt that this and other results summarized herein might lend insight to the problem.

II. The Dantzig Conjecture

Each component for the initial basic solution involving X is taken to be unity and also each component of Y in the terminal basic solution. This can always be done with no loss of generality by a change of units providing these basic solutions are (as assumed) nondegenerate. A mathematical statement of this conjecture follows.

Dantzig Conjecture: Let $\bar{B} = [b_{ij}]$ be a nonsingular square matrix whose rows sum to unity, and such that each column that has a positive element has a unique maximal element. Consider the program

$$(3) \qquad \begin{gathered} Y \geq 0 \,, \; X \geq 0 \\ IY + \bar{B}X = \bar{b} \end{gathered}$$

where $\bar{b}$ is a column vector whose components are all equal to 1. Let $[b_{ij}^{-1}] = \bar{B}^{-1}$. For all columns of $\bar{B}$ and $\bar{B}^{-1}$ that contain a strictly

positive element define

$$b_{k(j)j} = \operatorname*{Max}_i b_{ij}$$

$$b^{-1}_{\ell(j)j} = \operatorname*{Max}_i b^{-1}_{ij}$$

Then there exists a pair of variables x_j , $y_{k(j)}$ or $x_{\ell(j)}$, y_j such that the program obtained from (1) by deleting this pair and their corresponding columns is infeasible. Equivalently each feasible set of basic variables must contain at least one of this pair.

The reason for examining only these columns that have a positive element is that the only variables corresponding to these columns can be introduced into the basis at a finite value.

III. Using the Conjecture to Prove the Hirsch Conjecture

Throughout this section when a problem is stated in the form

$$IY + \bar{B}X = \bar{b}$$

it will be understood that $\bar{B} = [b_{ij}]$ is a nonsingular square matrix such that the sum of its columns is $\bar{b}$, a column vector whose elements are all 0 or 1. $I^{(m)}$, $\bar{B}^{(m)} = [b^{(m)}_{ij}]$, $\bar{b}^{(m)}$ are matrices containing m rows and having the same meaning as I , $\bar{B}$, and $\bar{b}$. Also b^{-1}_{ij} , $k(j)$, $\ell(j)$ have the same meanings as in the statement of the conjecture.

In addition the problems:

(4) $$X \geq 0$$

$$A_1X = b$$

and

(5) $$\bar{X} \geq 0$$
$$A_2\bar{X} = b$$

are equivalent if $A_2 = A_1 D$ where D is a diagonal matrix whose diagonal elements are all strictly positive. This happens if and only if the j^{th} column of A_2 is a positive multiple of the j^{th} column of A_1 for all j. It should be noted that if $\bar{X}$ solves (5) then $X = D\bar{X}$ solves (4) and therefore the set of feasible bases is identical for both problems. It follows that if the Hirsch Conjecture holds for a particular problem it holds for any problem equivalent to it in the above sense.

Lemma 1: If X and Y are feasible bases to the problem:

$$Y \geq 0 \ , \ X \geq 0$$
$$IY + BX = b$$

and Y is nondegenerate then there is an equivalent problem of the form:

$$\bar{Y} \geq 0 \ , \ \bar{X} \geq 0$$
$$I\bar{Y} + BX = \bar{b} \qquad \bar{b} = (1, 1, \ldots, 1)$$

Proof: Let $Y = b$, $X = 0$ and $Y = 0$, $X = d$ be the basic solutions corresponding to the bases X and Y respectively. Multiplying the j^{th} column of B by d_j and the j^{th} column of I by b_j if $b_j \neq 0$ yields an equivalent problem which has the corresponding basic solutions $Y = b'$, $X = 0$ and $Y = 0$, $X = d'$ where all components of b' are one or zero and all components of d' are one. Multiply the j^{th} row by $1/b_j \neq 0$ yields the desired form.

COROLLARY 2: If in lemma 1, all components of b are strictly

positive, then all elements of $\bar{b}$ are equal to 1.

Proof: Since all components of b are positive, it follows that the equivalent problem obtained in lemma 1 has basic solution $X = 0$, $Y = \bar{b}'$ where all components of b' are one. From this it follows that all components of $\bar{b}$ are one.

Lemma 3: Suppose in the problem of rank m:

$$\begin{aligned} Y^{(m)} \geq 0 \, , \; X^{(m)} \geq 0 \\ I^{(m)}Y^{(m)} + \bar{B}^{(m)}X^{(m)} = \bar{b}^{(m)} \end{aligned} \tag{6}$$

the Dantzig conjecture holds and let all components of $\bar{b}^{(m)}$ be 1, $k(j)$ be unique for all $b_{k(j)j} > 0$, and $\ell(j)$ unique for all $b^{-1}_{\ell(j)j} > 0$. Then the number of feasible pivot operations required to go from the canonical form with respect to $Y^{(m)}$ to that of $X^{(m)}$ is at most one more than the number required to go from the canonical form with respect to $Y^{(m-1)}$ to that of $X^{(m-1)}$ in a problem of rank $m - 1$ of the form:

$$\begin{aligned} Y^{(m-1)} \geq 0 \, , \; X^{(m-1)} \geq 0 \\ I^{(m-1)}Y^{(m-1)} + \bar{B}^{(m-1)}X^{(m-1)} = \bar{b}^{(m-1)} \end{aligned} \tag{7}$$

Proof: Suppose there is a pair x_j, $y_{k(j)}$ such that $b_{k(j)j} > 0$ and such that any feasible basis might contain at least one of these as a basic variable. Introduce x_j into the basis, dropping $y_{k(j)}$, and let row i be the row in which the coefficient of x_j is 1 in the resulting canonical form. It follows that the introduction of any sequence of variables, which does not include $y_{k(j)}$, into successive feasible bases cannot drop x_j from any of the bases. Deleting row

i and the columns corresponding to x_j and $y_{k(j)}$ yields a problem which is equivalent to one of the form (7). It follows that x_j and any sequence of feasible pivot operations on the above problem is a sequence of feasible pivots to the original problem and the theorem holds. If there is no pair x_j , $y_{k(j)}$ with the above mentioned properties, then since we assumed the Dantzig conjecture to hold in this case there exists a pair $x_{\ell(j)}$, y_j such that $b^{-1}_{\ell(j)j} > 0$ and any feasible basis includes at least one of these as a basic variable. In this case it follows that the number of feasible pivots required to go from the canonical form with respect to $X^{(m)}$ to that of $Y^{(m)}$ is at most one more than that required to go from $X^{(m-1)}$ to $Y^{(m-1)}$ in a problem of the form,

$$Y \geq 0 \,,\, X \geq 0$$
$$\bar{B}^{(m-1)}Y^{(m-1)} + I^{(m-1)}X^{(m-1)} = \bar{b}^{(m-1)}$$

which establishes lemma 3.

COROLLARY 4: Suppose in a problem of the form (6) all conditions of lemma 3 except the uniqueness property of $k(j)$ are met. Then the number of feasible pivot operations required to go from the canonical form with respect to $Y^{(m)}$ to that of $X^{(m)}$ is at most one more than that required to go from the canonical form with respect to $Y^{(m-1)}$ to that of $X^{(m-1)}$ in a problem of the form (7).

<u>Proof</u>: Replace $\bar{B}^{(m)}$ by $\bar{B}(\epsilon)^{(m)}$ where

$$\bar{b}_{ij}(\epsilon) = \bar{b}_{ij} + j\epsilon^{i} \qquad \text{for } j \neq m$$
$$= \bar{b}_{ij} - \sum_{k=1}^{m-1} k\epsilon^{i} \quad \text{for } j = m$$

For $\epsilon > 0$ sufficiently small,

$$\text{(8)} \qquad \begin{gathered} Y^{(m)} \geq 0 \;,\; X^{(m)} \geq 0 \\ I^{(m)}Y^{(m)} + \bar{B}^{(m)}(\epsilon)X^{(m)} = \bar{b}^{(m)} \end{gathered}$$

satisfies all the conditions of lemma 3 and any feasible basis to (8) is a feasible basis to the original problem.

COROLLARY 5: Suppose in a problem of the form (6) all components of $\bar{b}^{(m)}$ are 1. Then the number of feasible pivot operations required to go from the canonical form with respect to $Y^{(m)}$ to that of $X^{(m)}$ is at most one more than the number required to go from the canonical form with respect to $Y^{(m-1)}$ to that of $X^{(m-1)}$ in a problem of the form (7).

Proof: Replace $[\bar{B}^{(m)}]^{-1}$ by $[\bar{D}^{(m)}]^{-1}(\epsilon)$ where $[\bar{B}^{(m)}]^{-1}(\epsilon)$ has the same meaning as $\bar{B}^{(m)}(\epsilon)$ in the proof of corollary 4. For $\epsilon > 0$ sufficiently small,

$$\text{(9)} \qquad \begin{gathered} Y^{(m)} \geq 0 \;,\; X^{(m)} \geq 0 \\ [\bar{B}^{(m)}]^{-1}(\epsilon)Y^{(m)} + I^{(m)}X^{(m)} = \bar{b} \end{gathered}$$

satisfies all the conditions of corollary 4 and any feasible basis to 9 is a feasible basis to the original problem.

COROLLARY 6: The number of feasible pivot operations required to go from the canonical form with respect to the basis $Y^{(m)}$ to that of $X^{(m)}$ in any problem of the form (6) is at most one more than the number required to go from the canonical form with respect to $Y^{(m-1)}$ to that of $X^{(m-1)}$ in a problem of the form (7).

Proof: Replace all zero components of $\bar{b}^{(m)}$ by ϵ to obtain:

$$(10) \qquad \begin{aligned} & Y^{(m)} \geq 0 \ , \ X^{(m)} \geq 0 \\ & I^{(m)}Y^{(m)} + \bar{B}^{(m)}X^{(m)} = \bar{b}^{(m)}(\epsilon) \end{aligned}$$

For $\epsilon > 0$ sufficiently small $Y^{(m)}$ and $X^{(m)}$ are both feasible, nondegenerate basis to (10) and any feasible basis to (10) is a feasible basis to the original problem. By lemma 1 and corollary 2, there exists a program equivalent to (10) of the form (6) such that all components of $\bar{b}^{(m)}$ are 1 and the conditions of corollary 5 are satisfied.

THEOREM 7: Let X and Y be feasible bases to a linear programming problem of the form (1) of rank m^* and suppose the Dantzig conjecture holds for all $m \leq m^*$. If the basis X is nondegenerate then the program may be put into the form (2) by a sequence of m or less feasible pivot operations.

Proof: The theorem is trivially true for $m = 1$. For $m > 1$ the theorem follows from corollary 6 and induction.

IV. A Counter-Example

As was mentioned earlier, the Dantzig Conjecture is actually false in general. The following tables show a counter-example, which incidentally is not a counter-example to the Hirsch Conjecture, demonstrating that the two are not equivalent. The pairs $(x_j , y_{k(j)})$ are (x_1 , y_4) , (x_2 , y_6) , (x_3 , y_2) , (x_4 , y_2) , (x_5 , y_6) , (x_6 , y_4) and the pairs $(x_{\ell(j)} , y_j)$ are (x_3 , y_1) , (x_4 , y_2) , (x_4 , y_3) , (x_3 , y_4) , (x_4 , y_5) , (x_4 , y_6) . The sequence of six successive feasible pivots required to go from the basis Y to the basis X is $x_1 , x_2 , x_4 , x_5 , x_6 , x_3$.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1						.037037	.111111	.185185	.259260	.333333	.074074	=	1
	1					.086957	.173913	.260869	.347826	0	.130435	=	1
		1				.038461	.076923	.153846	.192308	.269231	.269231	=	1
			1			.285714	.214286	.107143	.071429	.035714	.285714	=	1
				1		.275862	.103448	.137931	.137931	.275862	.068966	=	1
					1	.181818	.227273	.045455	.090909	.363636	.090909	=	1

Table 1: The counter example with Y as the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
-13.101331	4.623872	4.706424	-6.050762	7.053425	3.768372	1						=	1
13.012808	-3.537300	-8.464069	8.277069	-7.514573	-.773935		1					=	1
53.138052	-20.926830	-24.833330	29.756046	-11.964016	-24.169922			1				=	1
-40.491148	18.605042	19.372819	-24.255224	10.824468	16.944042				1			=	1
3.627007	-2.521647	-.340534	.654862	-.135657	-.284031					1		=	1
-6.915629	1.540599	6.153491	-1.833603	.379855	1.675286						1	=	1

Table 2: The counter-example with X as the basis

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
		[illegible]	[illegible].000000			-.979593	-.[illegible]0614	[illegible]	.5[illegible]0[illegible]10	1		=	.244890
1		-1.[illegible]	1.0[illegible]70[illegible]			.280424	.22751[illegible]	.08[illegible]	.0[illegible]			=	.661[illegible]78
		-.[illegible]	[illegible].000000			1.1224[illegible]	.81[illegible]	[illegible]	.13[illegible]76			=	[illegible]
		-1.[illegible]58	1.090[illegible]		1	.[illegible]599[illegible]	.[illegible]012	-.0[illegible]	-.1187[illegible]			=	.595550
		-1.1[illegible]410	[illegible]	1		.[illegible]	.19[illegible]525	.052076	-.021112			=	.69317[illegible]
	1	.069210	-.[illegible]21[illegible]0			-.059[illegible]	.0674[illegible]	.215616	.[illegible]23868			=	.5[illegible]7470

Table [illegible]a: [illegible] basic solution with neither of [illegible] pair of variables ($x_{[illegible]}$, $y_{[illegible]}$) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
-1.421046					4.052628	.684209	.763159	-.078944		1	.263158	=	2.631582
5.684184					-5.210502	-.736838	-.552631	.815782	1		-.052631	=	.473682
-.710524		1			-.089072	-.004050	-.022268	.018219			.208502	=	.200404
-.355264			1		.227445	.313910	.226504	.051692			.280075	=	.872181
-.392013				1	-.399275	.188747	-.030854	.047187			.003330	=	.208711
-1.977107	1				1.812346	.343248	.356132	-.022881			.148742	=	.835241

Table 3b: A basic solution with neither of the pair of variables (x_2, y_6) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1	-.771994				.10185[illegible]	-.011574		-.011574		.370371	-.017362	=	.329861
	3.593749				-2.749995	-.187497		.812497	1	-.999997	.218751	=	.843754
	-.580570	1			.105770	.007211		.007211		.307693	.203125	=	.525240
	.051337		1		-.982141	.111607		.075892		-.[illegible]21428	.203125	=	.05919[illegible]
	-.34[illegible]98[illegible]			1	-.1[illegible]	.211607		.038793		.20[illegible]897	.00[illegible]466	=	.463363
	-1.437496				5.499990	.374997	1	-.12499[illegible]		1.999994	.312499	=	4.062494

Table 3c: A basic solution with neither of the pair of variables (x_3, y_2) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1	-.745372		.086421			-.003087			.00[illegible]17[illegible]	.[illegible]6419	.0015[illegible]	=	.[illegible]41049
	.392045		-1.378787		1	-.178030			.128787	.314[illegible]94	-.251893	=	.013258
	-.[illegible]463	1	.179485			.032051			-.0256[illegible]1	.275[illegible]41	.233975	=	.515025
	5.750018		-4.566674			-.833330		1	1.666670	-.166666	-.583330	=	1.083344
	-.495592		- .080457	1		.209771			-.0402[illegible]0	.272989	-.018677	=	.423851
	-2.875009		6.599997			1.749995	1		-.500000	.249996	1.624995	=	4.124988

Table 3d: A basic solution with neither of the pair of variables (x_4 , y_2) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1			.453702		-.916664			.192129	.20[illegible]4	.016204	.120370	=	.537038
	1		.456515		-1.1[illegible]			[illegible]	.271740	-.413476	.152173	=	.260870
		1	.201926		-.[illegible]			.151[illegible]	.1[illegible]54	.0[illegible]41[illegible]	.273[illegible]47	=	.[illegible]7307[illegible]
			3.750010		-3.[illegible]0011	1		.[illegible]	-.12[illegible]	-2.[illegible]87503	1.750000	=	.500000
			-1.689[illegible]60	1	1.1[illegible]			.00[illegible]21	.1[illegible]	.[illegible]	-.[illegible]10[illegible]5	=	.[illegible]827[illegible]
			-3.99999[illegible]		10.[illegible]		1	-.24999[illegible]	.[illegible]9999[illegible]	[illegible].74999[illegible]	-.99999[illegible]	=	3.99999[illegible]

Table [illegible]e: A basic solution with neither of the pair of variables (x_5, y_6) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1			-.129030				.003333	.171296	.250001	.320703	.037037	=	.870370
	1		-.304350				.108695	.220260	.320007	-.010870	.043470	=	.695650
		1	-.134014				.048077	.139423	.102093	.204423	.230770	=	.865386
			3.500004			1	.750002	.375001	.250002	.124999	1.000000	=	3.500004
			-.305510	1			-.103449	.034403	.068965	.241379	-.206896	=	.034482
			-.036364		1		.090909	-.022727	.045454	.340909	-.090909	=	.303636

Table 3f: A basic solution with neither of the pair of variables (x_6, y_6) in the basis.

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_0		
.599162		-1.343271	1		.347093	.319349	.250416	.027219				=	.602964
5.504829		.252425			-5.232906	-.737000	-.558252	.020381	1			=	.524269
-.379642		-.017411		1	-.397720	.160010	-.030406	.040870				=	.205222
-1.470232	1	-.713301			1.075890	.340137	.302010	-.035878				=	.692277
-.524669		-1.262134			4.165049	.689320	.791264	-.101939		1		=	2.378646
-3.407756		4.79[illegible]13			-.42[illegible]	-.019422	-.100798	.087379			1	=	.961159

Table 4a: A basic solution with neither of the pair of variables (x_3 , y_1) in the basis.

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_6		
	-.500006	-2.383074	1.974148	2.070212		1	.306921		-.205642			=	1.161279
1	-.240743	-.678936	.642208	-.620858			.089579		.004032			=	.093071
	4.500016	-.336074	-1.875454	.570798			.413924	1	1.445352			=	2.859285
	.522725	-.479112	.103195	-.863945	1		.214067		.068795			=	.282862
	-1.499999	1.985890	-1.645123	1.899828			-.204102		.004700	1		=	.740598
	-1.000006	2.260868	2.434792	-2.521739			.260872		-.086956		1		1.173915

Table 4b: A basic solution with neither of the pair of variables $(x_4\ ,\ y_2)$ in the basis.

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_6		
		-3.429777	2.502125		2.293017	1	.793619	-.155316	-.272339			=	1.305964
1		-.413067	.335302		-.043184		.004904	.120211	.145155			=	.278330
		.473071	-.345121	1	-1.074981		-.178431	.124071	.106529			=	.052969
	1	-.134009	-.372990		.130354		.114615	.206302	.307078			=	.628676
		.055104	-1.540930		2.240611		.246809	.072341	.263029	1		=	1.582979
		3.319140	1.191495		-2.574466		.074466	.521274	.409302		1	=	1.936169

Table 4c: A basic solution with neither of the pair of variables (x_4 , y_3) in the basis.

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_6		
		-3.429777	2.502125		2.293617	1	.793019	-.155316	-.272339			=	1.305964
1		-.413067	.335302		-.643184		.004964	.128211	.145155			=	.278330
		.473071	-.345121	1	-1.074981		-.178431	.124071	.100529			=	.052969
	1	-.134689	-.372990		.130354		.114615	.206302	.307678			=	.628676
		.885104	-1.540930		2.246011		.246609	.072341	.263829	1		=	1.582979
		3.319140	1.191495		-2.574466		-.074400	.521274	.409362		1	=	1.936169

Table 4d: A basic solution with neither of the pair of variables (x_3, y_4) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
-3.204430	.105972		-.156200	4.423733	-.340000	1			-.242940			=	.757000
-4.[illegible]7949	4.591324		-2.[illegible]0143	-2.765519	[illegible].622991		1		.430904			=	1.430904
1.2[illegible]	[illegible]514		-1.[illegible]	1.011400	-2.449950			1	1.281854			=	2.281864
-2.0[illegible]101	[illegible]		-1.[illegible]023	.55[illegible]	.074030				.051019			=	.051019
2.919255	-2.1[illegible]		[illegible]	[illegible]	.013010				.017570	1		=	1.017578
[illegible].945[illegible]	-.3[illegible]		[illegible]	-[illegible]	-3[illegible]				-.317[illegible]		1		.682365

Table 4: A basic solution with neither of the pair of variables (x_4, y_5) in the basis.

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_0		
-3.2[illegible]450	.103972		-.1[illegible]208	4.423733	-.348000	1			-.242940			=	.757000
[illegible]949	4.591324		-2.3[illegible]0143	-2.[illegible]05515	0.628991		1		.430904			=	1.430904
1.233[illegible]	2.922314		-1.335004	1.911[illegible]	-2.449950			1	1.231864			=	2.281864
-2.090101	.900360	1	-1.252023	.550745	.874030				.051619			=	.051619
2.915255	-2.194[illegible]		.22[illegible]504	.054015	.013010				.017578	1		=	1.017578
5.945707	-4.309019		5.[illegible]712	-3.050370	-3.700739				-.317035		1	=	.002305

Table 4f: A basic solution with neither of the pair of variables (x_4, y_0) in the basis.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1			-.129630				-.083333	.171296	.250001	.328703	.037037	=	.870370
	1		-.304350				.108695	.228260	.326087	-.010870	.043478	=	.695650
		1	-.134614				.040077	.139423	.182693	.264423	.230770	=	.865386
			3.500004			1	.750002	.375001	.250002	.124999	1.000000	=	3.500004
			-.965518	1			-.103449	.034483	.068965	.241379	-.206896	=	.034482
			-.030304		1		.090909	-.022727	.045454	.340909	-.090909	=	.363636

Table 5a: The basic solution obtained after one pivot, x_1 .

y_1	y_2	y_3	y_4	y_5	y_0	x_1	x_2	x_3	x_4	x_5	x_6		
1			.453702		-.916604			.192129	.208334	.016204	.120370	=	.537038
	1		.45651[illegible]		-1.195045			.255433	.271740	-.418470	.152173	=	.260870
		1	.201926		-.528848			.151442	.158054	.084104	.278847	=	.673078
			8.750010		-8.250011	1		.562498	-.124996	-2.087503	1.750000	=	.499999
			-1.689660	1	1.137938			.008021	.120689	.629313	-.310345	=	.448278
			-0.999993		10.999989		1	-.249996	.499996	3.749904	-.999998	=	3.999996

Table 5b: The basic solution obtained after two pivots, x_1 , x_2 .

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1	-.706070		.103700		.000001			-.003704		.337037	.003704	=	.337037
	3.079995		1.679973		-4.399967			.939993	1	-1.539988	.559995		.960001
	-.503847	1	-.004609		.169225			.002300		.328400	.190001		.520770
	.459984		0.900000		-0.799988	1		.079993		-2.879995	1.619997		.619996
	-.444135		-1.092414	1	1.060903		1	-.104026		.015172	-.377930		.332416
	-1.039981		-7.039972		13.199952			-.719988		4.519981	-1.279993		3.520000

Table 5c: The basic solutions obtained after three pivots, x_1, x_2, x_4

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1	-.583040		.006134	-.413455	-.690041			.039637			.159961	=	.199598
	2.310954		-1.395093	1.389156	-1.247030			.741960	1		-.153974	=	1.587987
	-.404890	1	.69[illegible]900	-.4029[illegible]	-.503256			.044546			.342282	=	.386828
	-1.109141		2.2[illegible]19	[illegible]2990	-2.903550	1		.309643			.484774	=	1.794418
	-.514836		-2.[illegible]21490	1.228[illegible]	2.047378			-.128594		1	-.463620	=	.407786
	.682667		2.[illegible]121	-5.544010	3.945844		1	-.138746			.315560	=	1.676814

Table 51: The basic solution obtained after four pivots, x_1 , x_2 , x_4 , x_5

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
1	-.[illegible]	-.[illegible]	.559976	-.2251[illegible]	-.[illegible]			.018819				=	.018819
	2.[illegible]	.[illegible]	-1.[illegible]1144	1.[illegible]	-1.[illegible]			.761999	1			=	1.761999
	-1.[illegible]	2.921[illegible]	2.0[illegible]	-1.[illegible]	-1.[illegible]			.130145			1	=	1.130145
	-.[illegible]	-1.[illegible]	1.2856 2	[illegible]	-2.1907[illegible]	1		.24655[illegible]				=	1.24655[illegible]
	-1.093253	1.[illegible]	-1.[illegible]	.[illegible]	1.365713			-.06325[illegible]		1		=	.931744
	1.58[illegible]	-2.[illegible]	.990206	-.[illegible]	5.144960		1	-.244887				=	.755113

Table 5e: The [illegible] solution [illegible] after five pivots, x_1, x_2, x_4, x_5, x_6.

y_1	y_2	y_3	y_4	y_5	y_6	x_1	x_2	x_3	x_4	x_5	x_6		
53.138052	-20.926830	-24.833330	29.756046	-11.964016	-24.169922			1				=	1.000000
-40.491148	16.605042	19.372819	-24.255224	10.824468	16.944042				1			=	1.000000
-6.915629	1.540599	6.153491	-1.833603	.379855	1.675286						1	=	1.000000
-13.101331	4.62[illegible]2	4.706424	-6.050762	7.053425	3.768372	1						=	1.000000
3.627007	-2.[illegible]	-.340534	.654862	-.135657	-.284031					1		=	1.000000
13.012808	-3.[illegible]7300	-8.464069	8.277069	-7.514573	-[illegible]		1					=	1.000000

Table 5f: The basic solution obtained after 6 pivots, x_1, x_2, x_4, x_5, x_6, x_3.